

SOME THEOREMS ABOUT THE RIESZ FRACTIONAL INTEGRAL

BY

NICOLAAS DU PLESSIS

I show in this paper that theorems which hold for Riemann-Liouville fractional integrals have analogues holding for the Riesz fractional integral [1]. Theorems 1, 2, and 3 are analogous to well-known results due to Hardy and Littlewood [2]. Theorem 4 is of a different character and is analogous to one recently proved by the author [3].

The Riesz fractional integral $f_\alpha(P)$ of order α is given by

$$f_\alpha(P) = K_m^{-1} \int_E r_{PQ}^{\alpha-m} f(Q) dQ, \quad \text{where} \quad K_m = \pi^{m/2} 2^\alpha \Gamma(\alpha/2) [\Gamma((m-\alpha)/2)]^{-1},$$

E denotes all of Euclidean m -space, and r_{PQ} denotes the distance between P and Q .

We assume always that $f(Q)$ is L -integrable over E .

I prove the following theorems.

THEOREM 1. *If $f(P) \in \text{Lip } \beta$, $0 < \beta < 1$ then $f_\alpha(P) \in \text{Lip } (\alpha + \beta)$, $0 < \alpha + \beta < 1$.*

THEOREM 2. *If $f(P) \in L^q$, $q > 1$, $1 + m/q > \alpha > m/q$, then*

$$f_\alpha(P) \in \text{lip } (\alpha - m/q).$$

THEOREM 3. *If $f(P) \in L^q$ and $0 < \alpha < m/q$, then*

$$f_\alpha(P) \in L^r, \quad \text{where} \quad \alpha = m(1/q - 1/r).$$

THEOREM 4. *If $f(P) \in L^q$ then*

(a) *for $0 < \alpha < m$, $2 < q < \infty$, $f_{\alpha/q}(P)$ is finite everywhere except possibly in a set which is of zero β -capacity for all $\beta > m - \alpha$;*

(b) *for $0 < \alpha < m$, $1 \leq q \leq 2$, $f_{\alpha/q}(P)$ is finite everywhere except possibly in a set of zero $(m - \alpha)$ -capacity.*

Both (a) and (b) are best possible.

1. Preliminaries. If P is the point $(x_1, \dots, x_m)$ and Q the point $(t_1, \dots, t_m)$ we define the points $(x_1 + t_1, \dots, x_m + t_m)$ and $(x_1 - t_1, \dots, x_m - t_m)$ to be $P + Q$ and $P - Q$ respectively. The distance $|P|$ of P from the origin $0 = (0, \dots, 0)$ is given by $|P|^2 = \sum_{r=1}^m x_r^2$, and $|P - Q|$ is the distance P to Q .

If, for $0 \leq \beta \leq 1$, $f(P + H) - f(P) = O(|H|^\beta)$ uniformly in P as $|H| \rightarrow 0$, we say that $f(P) \in \text{Lip } \beta$. If, in this, O is replaced by o we say that $f(P) \in \text{lip } \beta$.

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Next, we have

$$K_m(f_\alpha(P+H) - f_\alpha(P)) \\ = \left(\int_U + \int_{E-U} \right) (|Q-H|^{\alpha-m} - |Q|^{\alpha-m}) f(Q+P) dQ,$$

where U is the unit hypersphere having the origin as center. For $|H| < 1/2$ it is not difficult to establish that

$$|Q-H|^{\alpha-m} - |Q|^{\alpha-m} = O(|H|)$$

uniformly in $E-U$. The second integral is thus $O(H)$ uniformly in P , and so

$$K_m(f_\alpha(P+H) - f_\alpha(P)) \\ (2) \quad = \int_U (|Q-H|^{\alpha-m} - |Q|^{\alpha-m}) f(Q+P) dQ + O(|H|).$$

2. Proofs of Theorems 1 and 2. First, Theorem 1. The first term on the right-hand side of (1) of §1 may be rewritten in the form

$$(1) \quad \int_U (|Q-H|^{\alpha-m} - |Q|^{\alpha-m}) (f(Q+P) - f(P)) dQ \\ + f(P) \left\{ \int_{U'} |Q|^{\alpha-m} dQ - \int_U |Q|^{\alpha-m} dQ \right\},$$

where U' is the sphere U transforms into under the transformation $Q' = Q - H$. The expression in curly brackets is dominated by $\int_S |Q|^{\alpha-m} dQ$, where $S = U' + U - U'U$.

Now $mS < \pi^{m/2} [\Gamma((m+2)/2)]^{-1} \{ (1+|H|)^m - 1 \} = O(|H|)$ and $|Q|^{\alpha-m} < 2^{m-\alpha}$ in S for $|H| < 1/2$. Consequently, the second term in (2) is $O(|H|)$.

To deal with the first term we note that it is of order $\int_U |Q-H|^{\alpha-m} - |Q|^{\alpha-m} |Q|^\beta dQ$ and apply a uniform dilatation transformation of ratio $1:|H|$ and then a rotation which takes the transform of H into the point $1 = (1, 0, \dots, 0)$. The first term is then seen to be less than

$$|H|^{\alpha+\beta} \int_E ||Q-1|^{\alpha-m} - |Q|^{\alpha-m}| |Q|^\beta dQ = O(|H|^{\alpha+\beta}),$$

since it is again a simple matter to establish that the integral is finite. This proves Theorem 1.

Next, Theorem 2. Let $S(r)$ denote the hypersphere of radius r centered at the origin and write

$$A(\delta) = S(\delta) - S(|H|), \quad B(\delta) = U - S(\delta),$$

where δ will presently be defined. Split the right-hand side of (1) into integrals

I_1 over $S(|H|)$, I_2 over $A(\delta)$, and I_3 over $B(\delta)$. Then, firstly

$$\begin{aligned} |I_1| &\leq \left\{ \int_{S(|H|)} ||Q - H|^{\alpha-m} - |Q|^{\alpha-m}|^{q'} dQ \right\}^{1/q'} \\ &\quad \cdot \left\{ \int_{S(|H|)} |f(Q + P)|^q dQ \right\}^{1/q} \\ &= |H|^{\alpha-m/q} \left\{ \int_U ||Q - 1|^{\alpha-m} - |Q|^{\alpha-m}|^{q'} dQ \right\}^{1/q'} o(1) \end{aligned}$$

as $|H| \rightarrow 0$: we use the same transformation on the integral as before. Thus $I = o(|H|^{\alpha-m/q})$. Further

$$\begin{aligned} |I_2| &\leq \left\{ \int_{A(\delta)} ||Q - H|^{\alpha-m} - |Q|^{\alpha-m}|^{q'} dQ \right\}^{1/q'} \\ &\quad \cdot \left\{ \int_{A(\delta)} |f(Q + P)|^q dQ \right\}^{1/q}. \end{aligned}$$

It is again easy to show that, for $|H| < \delta/3$,

$$||Q - H|^{\alpha-m} - |Q|^{\alpha-m}| \leq C |H| |Q|^{\alpha-m-1}$$

and thus

$$|I_2| \leq C |H| \left\{ \int_{A(\delta)} |Q|^{(\alpha-m-1)q'} dQ \right\}^{1/q'} \left\{ \int_{A(\delta)} |f(Q + P)|^q dQ \right\}^{1/q}.$$

Further

$$\int_{A(\delta)} |Q|^{(\alpha-m-1)q'} dQ < |H|^{(\alpha-1)q' - m(q'-1)} \int_{E-U} |Q|^{(\alpha-m-1)q'} dQ,$$

so that

$$|I_2| \leq C |H|^{\alpha-m/q} \left\{ \int_{A(\delta)} |f(Q + P)|^q dQ \right\}^{1/q}.$$

Given any $\epsilon > 0$, we can choose δ so that $\int_{A(\delta)} |f(Q + P)|^q dQ$ is less than $(\epsilon/C)^q$, and so

$$|I_2| < \epsilon |H|^{\alpha-m/q}.$$

Finally

$$|I_3| \leq \left\{ \int_{B(\delta)} ||Q - H|^{\alpha-m} - |Q|^{\alpha-m}|^{q'} dQ \right\}^{1/q'} \left\{ \int_E |f(Q + P)|^q dQ \right\}^{1/q}.$$

For fixed δ , $|Q - H|^{\alpha-m} - |Q|^{\alpha-m} = O(|H|)$ uniformly in $B(\delta)$, and so $I_3 = O(|H|)$.

Thus, finally, $K_m(f_\alpha(P+H) - f_\alpha(P)) = o(|H|^{\alpha-m/q})$, giving the required result.

3. Proof of Theorem 3. We first prove a many-dimensional generalization of a theorem due to Hardy and Littlewood [2, Theorem 3].

LEMMA. *If $f(P) \in L^q$, $g(Q) \in L^r$, $1/q + 1/r > 1$, $q > 1$, $r > 1$ and $\mu = 2 - 1/q - 1/r$ then*

$$(1) \quad \int_E \int_E |Q - P|^{-m\mu} f(P) g(Q) dP dQ \leq K M_q(f) M_r(g),$$

where $M_q(f) = \{\int_E |f(P)|^q dP\}^{1/q}$ and $M_r(g)$ is similarly defined.

I prove here the case $m=3$, which is sufficiently typical.

Since an arithmetic mean is greater than the corresponding geometric mean we have

$$\begin{aligned} |P - Q|^2 &= (x_1 - t_1)^2 + (x_2 - t_2)^2 + (x_3 - t_3)^2 \\ &\geq 3 |x_1 - t_1|^{2/3} |x_2 - t_2|^{2/3} |x_3 - t_3|^{2/3} \end{aligned}$$

and so

$$|P - Q|^{-3\mu} \leq C |x_1 - t_1|^{-\mu} |x_2 - t_2|^{-\mu} |x_3 - t_3|^{-\mu}.$$

Consequently the left-hand side of (1) is not greater than a constant multiple of

$$(2) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x_1, x_2, x_3) g(t_1, t_2, t_3)}{|x_1 - t_1|^\mu |x_2 - t_2|^\mu |x_3 - t_3|^\mu} \cdot dt_3 dx_3 dt_2 dx_2 dt_1 dx_1.$$

By the Hardy-Littlewood theorem mentioned, which is the case $m=1$ of the lemma,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x_3 - t_3|^{-\mu} f(x_1, x_2, x_3) g(t_1, t_2, t_3) dt_3 dx_3$$

is dominated by $CF(x_1, x_2)G(t_1, t_2)$, where $F(x_1, x_2) = \{\int_{-\infty}^{\infty} |f(x_1, x_2, x_3)|^q dx_3\}^{1/q}$ and $G(t_1, t_2)$ is defined analogously.

Hence [2] is dominated by

$$C_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x_1 - t_1|^{-\mu} |x_2 - t_2|^{-\mu} F(x_1, x_2) G(t_1, t_2) dt_2 dx_2 dt_1 dx_1.$$

Applying the case $m=1$ of the lemma again to the inner two integrals we find that (1) is dominated by

$$C_1 C_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x_1 - t_1|^{-\alpha} F(x_1) G(t_1) dt_1 dx_1,$$

where $F(x_1) = \left\{ \int_{-\infty}^{\infty} |F(x_1, x_2)|^q dx_2 \right\}^{1/q}$ and $G(t_1)$ is defined analogously.

A final application of the lemma with $m=1$ shows that (1) is dominated by $C_1 C_2 C_3 M_q(F) M_r(G)$. Since $M_q(F)$ equals $\left\{ \int_E |f(P)|^q dP \right\}^{1/q}$ and a similar result holds for $M_r(G)$, we have the required result.

To prove Theorem 3 it is sufficient to prove that, for every $g(P)$ such that $M_{r'}(g) \leq 1$,

$$\int_E f_\alpha(P) g(P) dP \leq K M_q(f).$$

The left-hand side of this is equal to

$$(3) \quad K_m^{-1} \int_E \int_E |P - Q|^{\alpha-m} f(Q) g(P) dQ dP$$

and, since $\alpha - m = m(1/q - 1/r) - m = -m(2 - 1/q - 1/r')$, the lemma applies and shows (3) to be, in modulus, not greater than $K' M_q(f) M_{r'}(g) \leq K' M_q(f)$, thus proving the theorem.

4. Preliminaries about Theorem 4. We say, with Frostman [4, p. 26], that a non-negative additive set function $\mu(S)$ defined for all Borel sets in E is a distribution if $\mu(E) = 1$. Further, if $S \subseteq E$ and $\mu(S) = 1$ we say that the distribution is concentrated on S .

Let S be a given set. Suppose that there is a distribution concentrated on S such that

$$V_\beta = \sup_{P \in E} \int_E |Q - P|^{-\beta} d\mu(Q)$$

is finite. Then we say that S is of positive β -capacity. Otherwise S is said to be of zero β -capacity. Clearly, if S is of positive β -capacity, it is of positive γ -capacity for all $\gamma < \beta$. Further, if it is of zero β -capacity, it is of zero γ -capacity for all $\gamma > \beta$.

LEMMA. For $1 < q < 2$, and for every $\epsilon > 0$ for which $q - \epsilon > 1$, we have

$$(1) \quad \int_S \left\{ \int_E |Q - P|^{(\alpha/q')-m} d\mu(Q) \right\}^{q-\epsilon} dP \leq A(\alpha, \epsilon, m, q, S) V_{m-\alpha}^{(q-\epsilon)/(q-\epsilon)'},$$

where $A(\alpha, \epsilon, m, q, S)$ is a constant depending only on the parameters shown and S is a bounded set.

For $2 \leq q \leq \infty$ we have

$$(2) \quad \int_E \left\{ \int_E |Q - P|^{(\alpha/q')-m} d\mu(Q) \right\}^q dP \leq A(\alpha, m) V_{m-\alpha}^{q-1},$$

where $A(\alpha, m)$ is a constant depending only on the parameters shown.

We have

$$\begin{aligned} \left\{ \int_E |Q - P|^{(\alpha/q')-m} d\mu(Q) \right\}^{q-\epsilon} &= \left\{ \int_E |Q - P|^{-\alpha/q} |Q - P|^{\alpha-m} d\mu(Q) \right\}^{q-\epsilon} \\ &\leq \left\{ \int_E |Q - P|^{-\alpha(q-\epsilon)/q} |Q - P|^{\alpha-m} d\mu(Q) \right\} \\ &\quad \cdot \left\{ \int_E |Q - P|^{\alpha-m} d\mu(Q) \right\}^{(q-\epsilon)/(q-\epsilon)'} \end{aligned}$$

by Hölder's inequality. The second factor is not greater than $V^{(q-\epsilon)/(q-\epsilon)'}$ while the first is $\int_E |Q - P|^{\alpha\epsilon/q-m} d\mu(Q)$. The left-hand side of (1) is therefore not greater than

$$V_{m-\alpha}^{(q-\epsilon)/(q-\epsilon)'} \int_S dP \int_E |Q - P|^{\alpha\epsilon/q-m} d\mu(Q).$$

We invert the order of integration and note that

$$\int_S |Q - P|^{\alpha\epsilon/q-m} dP = A(\alpha, \epsilon, m, q, S), \text{ say.}$$

Furthermore $\int_E d\mu(Q) = 1$. (1) now follows.

To prove (2) I first show the result true for $q=2$ and then that this implies its truth for $q>2$. For this latter part of the proof I am indebted to Professor J. E. Littlewood.

We have first, on inverting the order of integration,

$$\begin{aligned} (3) \quad \int_E \left\{ \int_E |Q - P|^{(\alpha-m)/2} d\mu(Q) \right\}^2 dP \\ = \int_E \int_E \int_E |Q - P|^{(\alpha-m)/2} |R - P|^{(\alpha-m)/2} dP d\mu(Q) d\mu(R). \end{aligned}$$

To deal with the inner integral we dilate E uniformly, taking Q as the center of dilatation, in the ratio $1:|Q-R|$ and then rotate the dilated space so that the transform of $Q-R$ goes into the point 1. The inner integral then becomes

$$|Q - R|^{\alpha-m} \int_E |U|^{(\alpha-m)/2} |U+1|^{(\alpha-m)/2} dU = B(\alpha, m) |Q - R|^{\alpha-m}.$$

Consequently, the right-hand side of (3) is dominated by

$$B(\alpha, m) \int_E \int_E |Q - R|^{\alpha-m} d\mu(Q) \leq B(\alpha, m) V_{m-\alpha\mu}(E).$$

Since $\mu(E) = 1$ this gives the result for $q = 2$.

For $q > 2$, we have

$$\begin{aligned} \int_E \left\{ \int_E |Q - P|^{\alpha/q - m} d\mu(Q) \right\}^q dP \\ = \int_E \left\{ \int_E |Q - P|^{((q-2)/q)(\alpha-m)} |Q - P|^{(\alpha-2m)/q} d\mu(Q) \right\}^q dP \end{aligned}$$

and this, by Hölder's inequality, does not exceed

$$J = \int_E \left\{ \int_E |Q - P|^{\alpha-m} d\mu(Q) \right\}^{q-2} \left\{ \int_E |Q - P|^{(\alpha-m)/2} d\mu(Q) \right\}^2 dP.$$

The first curly bracket does not exceed $V_{m-\alpha}^{q-2}$ (by the definition of $V_{m-\alpha}$). So

$$J \leq V_{m-\alpha}^{q-2} \int_E \left\{ \int_E |Q - P|^{(\alpha-m)/2} d\mu(Q) \right\}^2 dP$$

and this, by the result for $q = 2$, does not exceed $V_{m-\alpha}^{q-2} B(\alpha, m) V_{m-\alpha}$. This gives the result for $q > 2$.

5. Proof of Theorem 4. Let

$$S_n(P) = \int_E |Q - P|^{\alpha/q - m} [f(Q)]_n dQ,$$

where

$$[f(Q)]_n = \begin{cases} |f(Q)| & \text{for } |f(Q)| \leq n \\ n & \text{for } |f(Q)| > n. \end{cases}$$

$S_n(P)$ is always defined and finite, and to prove the theorem it is sufficient to show that $S_n(P)$ is bounded everywhere except possibly in a set of zero β -capacity, where $\beta = m - \alpha$ for $1 \leq q \leq 2$ and $\beta > m - \alpha$ for $q > 2$.

Assume, then, that $S_n(P)$ is unbounded in a set M of positive β -capacity. It is then unbounded in a bounded set S of positive β -capacity. Then, first, there is a distribution concentrated on S such that $\int_E |Q - P|^{-\beta} d\mu(Q)$ is bounded for all P . Secondly, there is a function $n(P) \leq n$, taking only integer values such that $\int_S S_{n(P)}(P) d\mu(P)$ exists and is unbounded as $n \rightarrow \infty$. This is an adaptation of a known result used by Salem and Zygmund [5, embodied in the proof of Theorem II], but a proof is perhaps not unwelcome.

Let $\bar{S}_n(P) = \sup_{m \leq n} S_m(P)$ for $0 \leq m \leq n$. Then for all $P \in S$, $\{\bar{S}_n(P)\}^{-1} \rightarrow 0$ as $n \rightarrow \infty$. By Egoroff's theorem on uniform convergence it follows that there is a set $S' \subset S$ such that $\mu(S - S')$ is as small as we please, and in which $\{\bar{S}_n(P)\}^{-1} \rightarrow 0$ uniformly. It follows that, given any large number G , there is an integer $n_0 = n_0(G)$ such that, for all $P \in S'$, $\bar{S}_n(P) > G$ for all $n > n_0(G)$. Choose $n(P)$ such that $S_{n(P)}(P) = \bar{S}_n(P)$. Then

$$\int_S S_{n(P)}(P) d\mu(P) > G\mu(S') \quad \text{for } n > n_0,$$

and so

$$\int_S S_{n(P)}(P) d\mu(P) \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

I show this last to be impossible. We have

$$\begin{aligned} \left| \int_S S_{n(P)}(P) d\mu(P) \right| &= \left| \int_S \int_E |Q - P|^{\alpha/q-m} [f(Q)]_{n(P)} dQ d\mu(P) \right| \\ &\leq \int_E |f(Q)| \int_S |Q - P|^{\alpha/q-m} d\mu(P) dQ \end{aligned}$$

and this does not exceed $M_q(f) M_{q'} \left[\int_S |Q - P|^{\alpha/q-m} d\mu(P) \right]$. Now $M_q(f) < +\infty$ by hypothesis, and we have only to show that

$$(1) \quad M_{q'} \left[\int_E |Q - P|^{\alpha/q-m} d\mu(P) \right]$$

is bounded.

If $1 \leq q \leq 2$ then $q' \geq 2$ and (2) of the lemma of §4 immediately gives (1). If $q > 2$ we write $\beta = m - \gamma$. Since $\gamma < \alpha$ there is an $r < q$ such that $\alpha/q = \gamma/r$. We may suppose β so near $m - \alpha$ that $2 < r < q$ since the result, if true for a given β , is true for a larger β . We may now rewrite (1) in the form

$$M_{r'-\epsilon} \left[\int_E |P - Q|^{\gamma/r-m} d\mu(Q) \right],$$

which, since $r' > 2$, is shown to be bounded by invoking (1) of the lemma.

6. Theorem 4 is best possible. We show this by constructing a function $f(P) \in L^q$ and a set M of positive β -capacity (where $\beta = m - \alpha$ when $1 \leq q \leq 2$, and β is any number greater than $m - \alpha$ when $q > 2$) at every point of which $f_{\alpha/q}(P)$ is infinite. It will avoid unnecessary complication and fully illustrate the general procedure if this is done for the simplest case $m = 2$.

M is constructed as follows. Let $\{\xi_n\}$ be any sequence such that $0 < \xi_n < 1/2$. Let M_0 be the unit square $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$. From M_0 remove the rectangle $\xi_1 < x_1 < 1 - \xi_1$, $0 \leq x_2 \leq 1$ thus leaving the set M_1 . From the left-hand rectangle in M_1 remove the rectangle $\xi_1 \xi_2 < x_1 < \xi_1(1 - \xi_2)$, $0 \leq x_2 \leq 1$, and make a similar symmetric removal from the right-hand rectangle of M_1 , thus leaving a set consisting of 4 closed rectangles of length 1 and breadth $\xi_1 \xi_2$. If we continue in this manner we are left, after the n th removal, with a set M_n consisting of 2^n closed rectangles each of length 1 and breadth $\xi_1 \xi_2 \cdots \xi_n$. Consequently

$$mM_n = 2^n \xi_1 \xi_2 \cdots \xi_n.$$

It is known [5, p. 40] that the projection S of $M = \lim M_n$ on the x -axis will be of positive β -capacity if and only if

$$(1) \quad \sum_{n=1}^{\infty} 2^{-n} (\xi_1 \xi_2 \cdots \xi_n)^{-\beta} < \infty.$$

If S is of positive β -capacity there is a distribution ν concentrated on S such that $\int_0^1 |x_1 - t|^{-\beta} d\nu(t)$ is bounded for all x_1 . Let μ be an additive set function defined over E by

$$\mu(X) = \iint_X d\nu(x_1) dx_2.$$

Then

$$\int_M |P - Q|^{-\beta-1} d\mu(Q) = \int_0^1 \int_0^1 [(x_1 - t_1)^2 + (x_2 - t_2)^2]^{-(\beta+1)/2} dt_2 d\nu(t_1).$$

In the inner integral make the substitution $x_2 - t_2 = (x_1 - t_1)u$. It is then dominated by

$$|x_1 - t_1|^{-\beta} \int_{-\infty}^{\infty} (1 + u^2)^{-(\beta+1)/2} du = A(\beta) |x_1 - t_1|^{-\beta}.$$

Consequently, since μ is a distribution concentrated on M ,

$$\int_M |P - Q|^{-\beta-1} d\mu(Q) = \int_E |P - Q|^{-\beta-1} d\mu(Q) \leq A(\beta) \int_0^1 |x_1 - t_1|^{-\beta} d\nu(t_1),$$

which is bounded. Thus M is of positive $(\beta+1)$ -capacity if S is of positive β -capacity.

Define $\{f_n(P)\}$ over M_0 by

$$\begin{aligned} f_0(P) &= 0 \text{ in } M_0, \\ f_n(P) &= (\xi_1 \xi_2 \cdots \xi_n)^{-\alpha/q_n-1} \text{ in } M_n, \\ f_n(P) &= f_{n-1}(P) \text{ in } M_0 - M_n. \end{aligned}$$

Since $\{f_n(P)\}$ is, eventually, an increasing sequence of measurable functions the function $f(P)$ given by

$$f(P) = \lim_{n \rightarrow \infty} f_n(P) \text{ in } M_0,$$

$$f(P) = 0 \text{ in } E - M_0$$

exists and is measurable over E .

It is easily seen that, for $n = 1, 2, \dots$,

$$f(P) = 0 \text{ in } M_0 - M_1,$$

$$f(P) = (\xi_1 \xi_2 \cdots \xi_n)^{-\alpha/q} n^{-1} \text{ on } M_n - M_{n+1}$$

so that

$$\begin{aligned} \int_E |f(P)|^q dP &= \int_{M_0} |f(P)|^q dP = \sum_{n=1}^{\infty} \int_{M_n - M_{n+1}} |f(P)|^q dP \\ (2) \qquad &= \sum_{n=1}^{\infty} (\xi_1 \xi_2 \cdots \xi_n)^{-\alpha n - q} (m M_n - m M_{n+1}) \\ &= \sum_{n=1}^{\infty} (1 - 2\xi_{n+1}) 2^n (\xi_1 \xi_2 \cdots \xi_n)^{1-\alpha} n^{-q}. \end{aligned}$$

For $q > 2$, we may choose $\delta > 0$ so that $2(1+\delta) < q$, and then put

$$2\xi_n^{1-\alpha} = 1 + (1 + \delta)n^{-1}.$$

Then $2^{-n}(\xi_1 \xi_2 \cdots \xi_n)^{\alpha-1} \sim Cn^{-1-\delta}$ so that (1) with $\beta = 1 - \alpha$ is satisfied, showing S to be of positive $(1 - \alpha)$ -capacity, and hence that M is of positive $(2 - \alpha)$ -capacity.

Further, (2) is clearly finite, so that $f \in L^q$ over E .

Let $P(x_1, x_2)$ be any point of M . Let

$$M_n(P) = M_n \cdot S[t_2; x_2 - \epsilon_n \leq t_2 \leq x_2 + \epsilon_n], \quad \text{where } \epsilon_n = \xi_1 \xi_2 \cdots \xi_n / 2;$$

$$M_n^*(P) = (M_n - M_{n+1})S[t_2; x_2 - \delta_n \leq t_2 \leq x_2 + \delta_n],$$

$$\text{where } \delta_n = \xi_1 \xi_2 \cdots \xi_n (1 - 2\xi_{n+1}) / 2.$$

$M_n(P)$ then consists of 2^n squares each of side $\xi_1 \xi_2 \cdots \xi_n$, while $M_n^*(P) \subset M_n(P)$ and consists of 2^n squares each of side $\xi_1 \xi_2 \cdots \xi_n (1 - 2\xi_{n+1})$. No square in $M_n^*(P)$ contains P , but one of the squares, I_n (say), is contained in that one of the squares, J_n (say), of $M_n(P)$ which itself contains P . Furthermore, the I_n ($n = 1, 2, \dots$) are disjoint.

Now $|Q - P| < 2^{1/2} \xi_1 \cdots \xi_n$ for Q in J_n , and so certainly for Q in I_n , and thus

$$K_2 f_{\alpha/q}(P) = \int_{M_0} |Q - P|^{\alpha/q-2} f(Q) dQ = \sum_{n=1}^{\infty} \int_{M_n - M_{n+1}} \geq \sum_{n=1}^{\infty} \int_{I_n}.$$

This last is not less than

$$\begin{aligned} \sum_{n=1}^{\infty} (2^{1/2} \xi_1 \cdots \xi_n)^{\alpha/q-2} (\xi_1 \cdots \xi_n)^{-\alpha/q} n^{-1} (\xi_1 \cdots \xi_n)^2 (1 - 2\xi_{n+1})^2 \\ = 2^{\alpha/2q-1} \sum_{n=1}^{\infty} (1 - 2\xi_{n+1})^2 n^{-1} = +\infty. \end{aligned}$$

Consequently, $f_{\alpha/q}(P)$ is infinite at every point of M , giving the required example in the case of $q > 2$, thus showing part (a) of Theorem 4 best possible.

For the case $q \leq 2$, let β be any positive number less than $1 - \alpha$ and let ξ be such that $2\xi^{(1-\alpha+\beta)/2} = 1$. Consider the set M with $\xi_n = \xi$ for all n . Since $2\xi^\beta > 1$, M is of positive $(\beta+1)$ -capacity. Defining $f(P)$ as before, we use exactly the same argument to show that $f_{\alpha/q}(P) = +\infty$ at every point of M . Furthermore, since $2\xi^{1-\alpha} < 1$, (2) is bounded, so that $f \in L^q$.

This shows part (b) of Theorem 4 best possible.

7. The lemma of §4 is best possible. Consider, e.g., (2) of the lemma. Suppose this is not the case, i.e. that there is an $\epsilon > 0$ for which, in general,

$$M_{q+\epsilon} \left[\int_E |Q - P|^{\alpha/q' - m} d\mu(Q) \right] < \infty.$$

If, then, $f(P) \in L^{(q+\epsilon)'}$ we may say that

$$\left| \int_E S_{n(P)}(P) d\mu(P) \right| \leq M_{(q+\epsilon)'}(f) M_{q+\epsilon} \left[\int_E |Q - P|^{\alpha/q' - m} d\mu(Q) \right]$$

which is bounded. This would imply that (b) of Theorem 4 is not best possible. Since it is best possible we have shown (2) best possible. A similar argument using (a) would show (1) best possible.

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RHODES UNIVERSITY,
GRAHAMSTOWN, SOUTH AFRICA